Elliptic curves over finite fields and point counting

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Introduction: The Discrete Logarithm Problem (DLP)

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Definition (Discrete Logarithm Problem)

Given g and g^k, the Discrete Logarithm Problem consists in finding $k \in \mathbb{Z}/n\mathbb{Z}$.

Attacks on the DLP

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Notation: $n = \#G$ and p is the larger prime factor of n

- The time complexity of the best generic attacks on the DLP is $O(\sqrt{p} \cdot \mathrm{poly}(\log(n)))$.
- There are specific attacks for $G = (\mathbb{Z}/p\mathbb{Z})^*$, but not for elliptic curves ...

Summary

1. Elliptic curves
   - Definition and the group law
   - One application to cryptography
2. Point counting methods for elliptic curves over finite fields
   - Schoof's algorithm
   - The SEA algorithm

Elliptic curves: definition

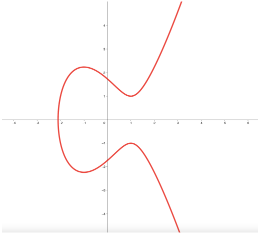
Let k be a field of characteristic p.

Definition

An elliptic curve E over k is given by the solutions $(x, y) \in k^2$ of an equation of the form

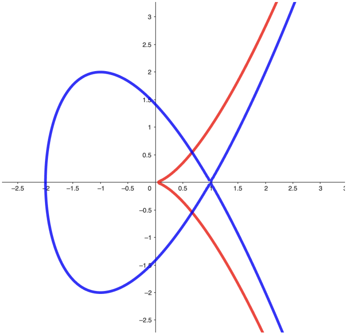
$$y^2 = x^3 + ax + b$$

with an additional point O called the "point at infinity", where $(a, b) \in k^2$ satisfy $4a^3 + 27b^2 \neq 0$. We denote by $E(k)$ the set of points of E.



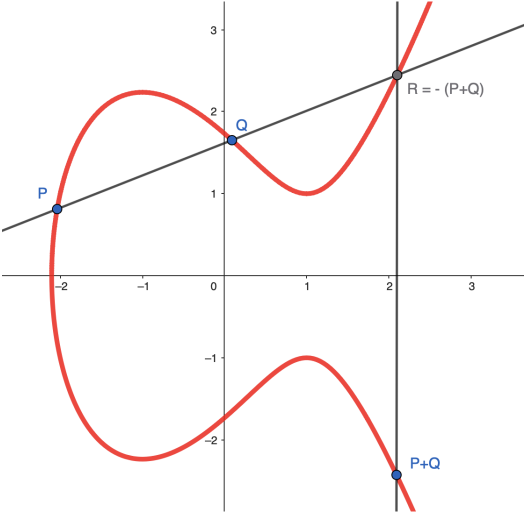
Singularity

The condition $4a^3 + 27b^2 \neq 0$ ensures that the curve is smooth.



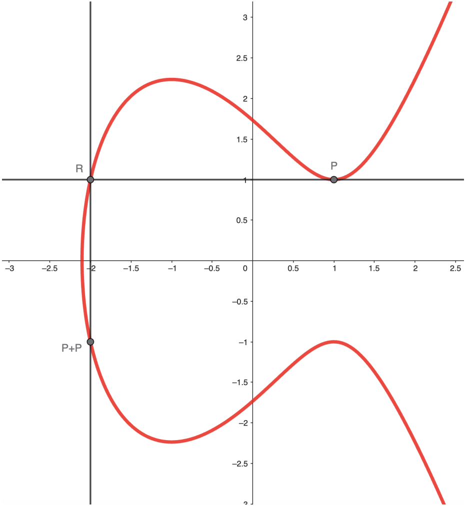
$y^2 = x^3$ (cusp); $y^2 = x^3 - 3x + 2$ (node)

The group law



Thus, elliptic curves are both algebraic and geometric objects.

Duplication of a point



Morphisms of elliptic curves: isogenies

Let E, E' be two elliptic curves over k. An isogeny $\phi : E \to E'$ is a map which respects the algebraic and geometric structures of E and E'. Concretely, it is a morphism of algebraic varieties such that

$$\phi(P + Q) = \phi(P) + \phi(Q)$$

for every points $P, Q \in E$.

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Definition/Proposition (torsion)

Let $m > 0$ be an integer. The m-torsion subgroup of E is

$$E[m] := \{P \in E(\overline{k}) : [m]P = O\}.$$

If m is coprime to $\mathrm{char}(k)$, then $E[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

The Diffie-Hellman protocol (1976)

Let $k = \mathbb{F}_q$ be a finite field, and assume that $E(\mathbb{F}_q)$ is cyclic of order n, generated by a point P.

Alice		Bob
chooses a random $a \in \mathbb{Z}/n\mathbb{Z}$	$\xrightarrow{[a]P}$	chooses a random $b \in \mathbb{Z}/n\mathbb{Z}$
computes $[a]P$	$\xleftarrow{[b]P}$	computes $[b]P$
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$\rightsquigarrow$ We have to find elliptic curves such that $\#E(\mathbb{F}_q)$ is a large prime.

The endomorphism of Frobenius

Let E be an elliptic curve over $\mathbb{F}_q$ given by a Weierstrass equation

$$y^2 = x^3 + ax + b.$$

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We have $\#E(\mathbb{F}_q) = q + 1 - t_E$. Hasse bound : $|t_E| \leq 2\sqrt{q}$.

Schoof's algorithm (1985)

Time complexity: $\widetilde{O}(\log(q)^5)$ (quite impossible to use in prcatice for cryptographic sizes...)

- The main idea of Schoof's algorithm: compute t_E modulo small primes $\ell \leq \ell_{max}$ such that

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How to compute t_E mod ℓ ?

- Evaluating the characteristic equation of Frobenius at $P = (x, y) \in E[\ell]$, we get $(x^{q^2}, y^{q^2}) - [t_E](x^q, y^q) + [q](x, y) = 0$.

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- We test whether the equality $(x^{q^2}, y^{q^2}) - [k](x^q, y^q) + [q_\ell](x, y) = 0$ is satisfied for $k = 0, \dots, \ell - 1$. The only k such that the last equality holds is n_ℓ.

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Fast exponentiation

We know that $E[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^2$. So, the x-coordinates of the ℓ-torsion points of E are the roots of a polynomial ψ_ℓ of degree $\frac{\ell^2-1}{2}$. Computations are performed in the ring

$$R_\ell = \frac{\mathbb{F}_q[x,y]}{(\psi_\ell(x), y^2 - x^3 - ax - b)}.$$

The SEA algorithm (90s)

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Definition

A prime $\ell \neq \mathrm{char}(\mathbb{F}_q)$ is said to be Elkies for E if and only if

$$\left(\frac{t_E^2 - 4q}{\ell}\right) = 0 \text{ or } 1.$$

Otherwise, it is said to be Atkin.

Heuristic: The number of Elkies and Atkin primes is approximately the same. If true, the complexity of the SEA algorithm is $\widetilde{O}(\log(q)^4)$.

Average results for the distribution of Elkies primes

Shparlinski and Sutherland have shown that the number of Elkies and Atkin primes is roughly the same, in average, over these two families:

- All elliptic curves defined over a finite field $\mathbb{F}_q$, when $q \to +\infty$

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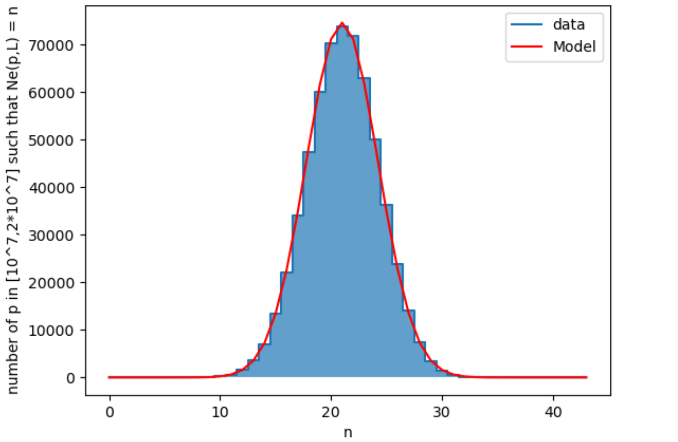
- Reductions modulo p of a given elliptic curve E defined over $\mathbb{Q}$

The distribution of Elkies primes is Gaussian

For an elliptic curve $E/\mathbb{Q}$, let $N_e(p, L)$ be the number of Elkies primes in $(L, 2L)$ for the elliptic curve E mod p.



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- Elkies primes have also been generalized
- Results about the distribution of Elkies primes are (partially) generalized

Questions ?

Thank you !